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6.1 EXPERIMENTS, OUTCOMES, SAMPLE SPACES, and EVENTS

Definitions: Experiment and Outcome

An ***experiment*** is an activity with an observable result.

Each possible result is called an ***outcome*** of the experiment.

Experiment 1: Flip a coin   Trial: One coin flip   Outcome: Heads

Experiment 2: Allow a conditioned rat to run a maze containing three possible paths
Trial: One run   Outcome: Path 1

Experiment 3: Tabulate the amount of rainfall in New York, NY in a year
Trial: One year   Outcome: 37.23 in.

Definition: Sample Space
The set of all possible outcomes of an experiment is called the ***sample space*** of the experiment.

So each outcome is an element of the sample space.

Example 1 Sample Space
An experiment consists of throwing two dice, one red and one green, and observing the numbers on the uppermost face on each. What is the sample space S of this experiment?

Solution:
Each outcome of the experiment can be regarded as an ordered pair of numbers, the first representing the number on the red die and the second the number on the green die.

S= {

<u>Definition:</u> Event

An ***<u>event</u>*** *E* is a subset of the sample space.

We say that the event ***occurs*** when the outcome of the experiment is an element of *E*.

<u>Example 2</u> Event

For the experiment of rolling two dice, describe the events:

a. E_1 = {The sum of the numbers is greater than 9}

b. E_2 = {The sum of the numbers is 7 or 11}.

<u>Definition:</u> Special Events

Let *S* be the sample space of an experiment.

The event corresponding to the empty set, ♓ is called the ***<u>impossible event</u>***, since it can never occur.

The event corresponding to the sample space itself, *S*, is called the ***<u>certain event</u>*** because the outcome must be in *S*.

<u>Events as Sets</u>

Let *E* and *F* be two events of the sample space *S*.

The event where *either* E *or* F or both occurs is designated by $E \cup F$.

The event where *both* E *and* F occur is designated by $E \cap F$.

The event where E does not occur is designated by E'.

Definition: **Special Events**

Let E and F be events in a sample space S. Then E and F are ***mutually exclusive*** (or *disjoint*) if $E \cap F = \emptyset$.

If E and F are mutually exclusive, then E and F cannot simultaneously occur; if E occurs, then F does not; and if F occurs, then E does not.

Example 3 **Mutually Exclusive Events**

For the experiment of rolling two dice, which of the following events are mutually exclusive?
E_1 = "The sum of the dots is greater than 9"
E_2 = "The sum of the dots is 7 or 11"
E_3 = "The dots on the two dice are equal"

Find $E_1 \cap E_2$, $E_1 \cap E_3$, and $E_2 \cap E_3$.

Solution:

E_1 =

E_2 =

E_3 =

$E_1 \cap E_2 =$

$E_1 \cap E_3 =$

$E_2 \cap E_3 =$

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6.2 ASSIGNMENTS OF PROBABILITIES

Definition: Probability of an Outcome

Let a sample space S consist of a finite number of outcomes $s_1, s_2, \ldots, s_N$.

To each outcome we associate a number, called the ***probability of the outcome***, which represents the relative likelihood that the outcome will occur.

A chart showing the outcomes and the assigned probability is called the ***probability distribution*** for the experiment.

Example 1 Probability of an Outcome with Probability Distribution
Toss an unbiased coin and observe the side that faces upward. Determine the probability distribution for this experiment.

Solution:
Since the coin is unbiased, each outcome is equally likely to occur.

Definition: Experimental Probability

Let a sample space S consist of a finite number of outcomes $s_1, s_2, \ldots, s_N$.

The ***relative frequency***, or ***experimental probability***, of each outcome is calculated after many trials.

The experimental probability could be different for a different set of trials and different from the probability of the events.

Fundamental Properties of Probabilities:
Let an experiment have outcomes $s_1, s_2, \ldots, s_N$ with respective probabilities $p_1, p_2, \ldots, p_N$. Then the numbers $p_1, p_2, \ldots, p_N$ must satisfy two basic properties:

1. Each of the numbers $p_1, p_2, \ldots, p_N$ is between 0 and 1

2. $p_1 + p_2 + \ldots + p_N = 1$

<u>Example 2</u> Experimental Probability

Traffic engineers measure the volume of traffic on a major highway during the rush hour.

a. Generate a probability distribution using the data generated over 300 consecutive weekdays.
b. Verify the fundamental properties for the probability distribution.
c. Determine the probability that 4000 cars are on the highway during rush hour.
d. Determine the probability that 6000 or more cars are on the highway during rush hour.

Number of cars observed	Frequency observed
≤ 1000	30
1001–3000	45
3001–5000	135
5001–7000	75
> 7000	15

<u>Addition Principle:</u>
Suppose that an event E consists of the finite number of outcomes $s, t, u, \ldots, z$.
That is $E = \{s, t, u, \ldots, z\}$.

Then, $P(E) = P(s) + P(t) + P(u) + \ldots + P(z)$,
where $P(A)$ is the probability of event A.

<u>Example 3</u>
Suppose that we toss a red die and a green die and observe the numbers on the sides that face upward.

a. Calculate the probabilities of the elementary events.

b. Calculate the probability that the two dice show the same number.

<u>Definition:</u> Inclusion-Exclusion Principle

Let E and F be any events. Then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F).$$

If E and F are mutually exclusive, then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F).$$

<u>Example 4</u> Inclusion-Exclusion Principle
A factory needs two raw materials. The probability of not having an adequate supply of material A is .05 and the probability of not having an adequate supply of material *B* is .03. A study determines that the probability of a shortage of both materials is .01. What proportion of the time will the factory not be able to operate from lack of materials?

<u>Converting between Odds and Probabilities:</u>

If P(*E*) = *p*, then the odds in favor of *E* are found by reducing the fraction $\frac{p}{1-p}$ to the form $\frac{a}{b}$, where *a* and *b* are integers having no common divisor.

Then the odds in favor of *E* are a to b.

<u>Example 5</u> Odds vs. Probability

a. Suppose the odds of rain tomorrow are 5 to 3. What is the probability that rain will occur?

b. The probability of obtaining a sum of 8 or more on a pair of dice is $\frac{15}{36}$.
What are the odds of obtaining a sum of 8 or more on a pair of dice?

<u>Example 6</u> Odds vs. Probability

You have entered your name into a drawing to win one of 25 prizes. There are 13 prizes for a free drink at the local coffee shop, 6 prizes for a free meal at a local diner, and 1 $50 gift card.

Determine the odds and probability of each.

EVENT	ODDS	PROBABILITY
Win the gift card.		
Win a coffee drink.		
Win a free meal.		

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6.3 CALCULATING PROBABILITIES OF EVENTS

<u>**Definition:**</u> **Probability of Equally Likely Outcomes**

Let *S* be a sample space consisting of *N* equally likely outcomes. Let *E* be any event.

Then, $Pr(E) = \frac{\#\ of\ outcomes\ in\ E}{N}$

<u>**Example 1**</u> **Equally Likely Outcomes**
An urn contains eight white marbles and two green marbles. A sample of three marbles is selected at random. Determine the probability of selecting only white marbles.

a. How many ways can you select a white marble?

b. How many ways can you select any color marble?

c. What is the probability of selecting only white marbles?

<u>**Example 2**</u> **Equally Likely Outcomes**
An urn contains eight white marbles and two green marbles. A sample of three marbles is selected at random. What is the probability that the sample contains at least one green marble?

<u>Example 3</u> Equally Likely Outcomes

Suppose that a cruise ship returns to the US from the Far East. Unknown to anyone, 4 of its 600 passengers have contracted a rare disease. Suppose that the Public Health Service screens 20 passengers, selected at random, to see whether the disease is present aboard ship.
What is the probability that the presence of the disease will escape detection?

> **<u>Definition:</u> The Complement Rule**
>
> Let E be any event, E' its complement.
>
> Then, $P(E) = 1 - P(E')$

<u>Example 4</u> Complement Rule

A group of 5 people is to be selected at random. What is the probability that 2 or more of them have the same birthday?

<u>Example 5</u>

A die is rolled five times. What is the probability of obtaining exactly three 4s?

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6.4 CONDITIONAL PROBABILITY AND INDEPENDENCE

DEFINITION: Conditional Probability

Let E and F be events is a sample space S. The *conditional probability* $\Pr(E \mid F)$ is the probability of event E occurring given the condition that event F has occurred. In calculating this probability, the sample space is restricted to F.

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

provided that $\Pr(F) \neq 0$.

PRODUCT RULE:

Product Rule If $\Pr(F) \neq 0$, then $\Pr(E \cap F) = \Pr(F) \cdot \Pr(E \mid F)$.

The product rule can be extended to three events,

$$\Pr(E1 \cap E2 \cap E3) = \Pr(E1) \cdot \Pr(E2 \mid E1) \cdot \Pr(E3 \mid E1 \cap E2)$$

Example 1

A sequence of two playing cards is drawn at random (without replacement) from a standard deck of 52 cards. What is the probability that the first card is red and the second is black?

Rephrase:

Solution:

Let E = "the second card is black," and F = "the first card is red."

<u>Definition:</u> Conditional Probability – Equally Likely Outcomes

Conditional Probability in Case of Equally Likely Outcomes

$$\Pr(E \mid F) = \frac{[\text{number of outcomes in } E \cap F]}{[\text{number of outcomes in } F]}$$

provided that [number of outcomes in F] $\neq 0$

<u>Example 2</u> Conditional Probability

Twenty percent of the employees of Acme Steel Company are college graduates. Of all its employees, 25% earn more than $50,000 per year, and 15% are college graduates earning more than $50,000. What is the probability that an employee selected at random earns more than $50,000 per year, given that he or she is a college graduate?

<u>Example 3</u> Conditional Probability

A sample of two marbles are selected from an urn containing 8 white marbles and 2 green marbles. What is the probability that the second marble selected is white given that the first marble selected was white?

<u>Definition:</u> Independence

Let E and F be events. We say that E and F are *independent* provided that

$$\Pr(E \cap F) = \Pr(E) \cdot \Pr(F).$$

Equivalently, they are independent provided that

$$\Pr(E \mid F) = \Pr(E) \text{ and } \Pr(F \mid E) = \Pr(F).$$

<u>Example 4</u> Independence

Let an experiment consist of observing the results of drawing two consecutive cards from a 52-card deck. Let E = "second card is black" and F = "first card is red".
Are these two events independent?

<u>Definition:</u> Independence of a Set of Events

A set of events is said to be *independent* if, for each collection of events chosen from them, say $E_1, E_2, \ldots, E_n$, we have

$$\Pr(E_1 \cap E_2 \ldots \cap E_n) = \Pr(E_1) \cdot \Pr(E_2) \cdot \ldots \cdot \Pr(E_n).$$

<u>Example 5</u> Independence of a Set

A company manufactures stereo components. Experience shows that defects in manufacture are independent of one another. Quality control studies reveal that:

> 2% of CD players are defective, 3% of amplifiers are defective, and 7% of speakers are defective.

A system consists of a CD player, an amplifier, and 2 speakers. What is the probability that the system is not defective?

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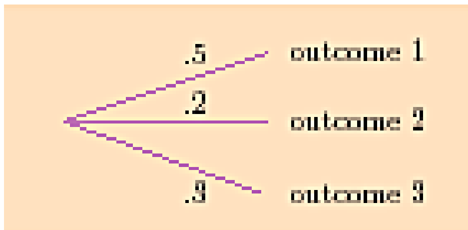
6.5 TREE DIAGRAMS

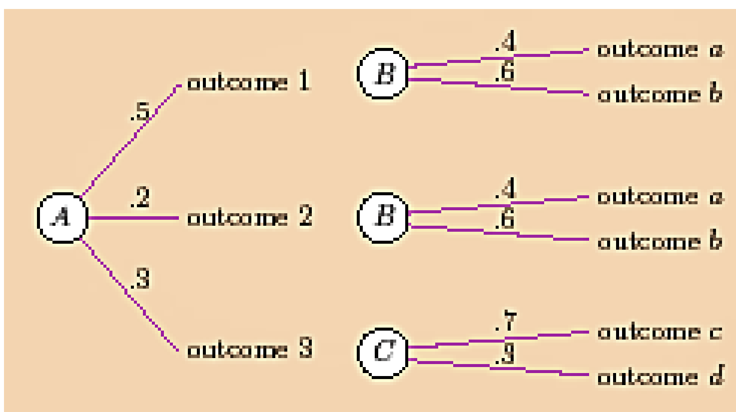
<u>Definition:</u> Tree Diagram

A ***tree diagram*** helps us represent the various events and their associated probabilities. The various outcomes of each experiment are represented as branches emanating from a point. Each branch is labeled with the probability of the associated outcome.



We represent experiments performed one after another by stringing together diagrams of the sort given in the previous slide. The probabilities for the second set of branches are conditional probabilities given the outcome from which the branches are emanating.

For example:



<u>Example 1</u> Using the given example above, determine each probability.

a. $\Pr(1, b)$

b. $\Pr(3, c)$

c. $\Pr(2, a)$

d. $\Pr(1, d)$

<u>Example 2</u> IRS Audit

This year experts predict that 20% of all taxpayers will file an incorrect tax return.
The Internal Revenue Service (IRS) itself is not perfect. IRS auditors claim there is an error when no problem exists about 10% of the time. The audits also indicate no error with a tax return when in fact there really is a problem 30% of the time.

The IRS has just notified a taxpayer that there is an error in his return. What is the probability that the return actually has an error?

<u>Example 3</u> Quality Control

A box contains 5 good light bulbs and 2 defective ones. Bulbs are selected one at a time (without replacement) until a good bulb is found. Find the probability that the number of bulbs selected is (*i*) 1, (*ii*) 2, (*iii*) 3.

<u>Example 4</u> Medical

The reliability of a skin test for active pulmonary tuberculosis (TB) is as follows:
Of people with TB, 98% have a positive reaction and 2% have a negative reaction;
of people free of TB, 99% have a negative reaction and 1% have a positive reaction.
From a large population of which 2 per 10,000 persons have TB, a person is selected at random and given a skin test, which turns out positive.
What is the probability that the person has active TB?