MATH 1500/MATH 1552
5.1 SETS

> **<u>Definitions: Set and Elements</u>**
> A ***<u>set</u>*** is any collection of objects. The objects, which may be countries, cities, years, numbers, letters, or anything else, are called the ***<u>elements</u>*** of the set. A set is often specified by a listing of its elements inside a pair of braces. A set may also be specified by giving a description of its elements.

<u>Example 1</u> State the given set.

1. The set of the first six letters of the alphabet. ______________________________

2. The set of even numbers between 1 and 11.______________________________

3. The set of ordered pairs for the graph $\{(a,b)$ where $b = a^2\}$.______________________

<u>Example 2</u>

Let A = {years from 1998 to 2011 in which unemployment is at least 5%}.
Let B = {years from 1998 to 2011 in which inflation is at least 3%}.

TABLE 2

Year	Unemployment (%)	Inflation (%)
1998	4.5	1.6
1999	4.2	2.2
2000	4.0	3.4
2001	4.7	2.8
2002	5.8	1.6
2003	6.0	2.3
2004	5.5	2.7
2005	5.1	3.4
2006	4.6	3.2
2007	4.6	2.8
2008	5.8	3.8
2009	9.3	−0.4
2010	9.6	1.6
2011	9.0	3.2

Using the table, the two sets are:

A = ____

B = ____

Definitions: Union and Intersection
The ***union*** of *A* and *B*, denoted *A U B*, is the set consisting of all those elements that belong to *either* *A* or *B* or both.

The ***intersection*** of *A* and *B*, denoted $A \cap B$, is the set consisting of those elements that belong to *both* *A* and *B*.

Example 3
From the previous example, let
A = {2002, 2003, 2004, 2005, 2008, 2009, 2010, 2011} and B = {2000, 2005, 2006, 2008, 2011}

a) Find A U B

b) Find $A \cap B$

Definitions: Subset and Empty Set
A set *B* is called a ***subset*** of *A* provided that every element of *B* is also an element of *A*.

The set that contains no elements at all is the ***empty set*** (or ***null set***) and is written as Ø.
The empty set is a subset of every set.

Example 4 List all possible subsets of {*a, b, c*}.

Definitions: Universal Set and Complement
The set *U* that contains all the elements of the sets being discussed is called a *universal set* (for the particular problem).

If *A* is a subset of *U*, the set of elements in *U* that are not in *A* is called the *complement* of *A*, denoted by *A'*.

Example 5 Let *U* = {*a,b,c,d,e,f,g*}, *S* = {*a,b,c*} and *T* = {*a,c,d*}.

a) Find S'

b) Find T'

c) Find $(S \cap T)'$

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5.2 A FUNDAMENTAL PRINCIPLE OF COUNTING

Definition: Number of Elements in A
If S is any set, we will denote the number of elements in S by $n(S)$.

For example,
if $S = \{1,7,11\}$, then $n(S) = 3$;

if $S = \emptyset$, then $n(S) = 0$.

Inclusion –Exclusion Principle
If S is any set, we will denote the number of elements in S by $n(S)$.

Let S and T be sets
Then
$$n(S \cup T) = n(S) + n(T) - n(S \cap T).$$

Example 1
In the year 2000, *Executive* magazine surveyed the presidents of the 500 largest corporations in the US. Of these 500 people, 310 had degrees (of any sort) in business, 238 had undergraduate degrees in business, and 184 had graduate degrees in business.
How many presidents had both undergraduate and graduate degrees in business?

Solution:
Let S = {presidents with an undergraduate degree in business}
Let T = {presidents with a graduate degree in business}.

So, $S \cup T$ = {presidents with degrees (of any sort) in business}
and $S \cap T$ = {presidents with both undergraduate and graduate degrees in business}

Determine, $n(S) =$ $n(T) =$ $n(S \cup T) =$

Find $n(S \cap T)$

Definition: Venn Diagram
A *Venn diagram* is a drawing that represents sets geometrically.

To construct a Venn diagram, draw a rectangle and view its points as the elements of *U*. Then draw a circle inside the rectangle for each set. The circles should overlap. View the points inside the circles as elements of each set.

Example 2

a. Draw a Venn diagram with three sets, *R, S* and *T*.
Shade the area that represents $R \cap S \cap T$

b. Draw a Venn diagram with three sets, *R, S* and *T*.
Shade the area that represents $R \cup S \cup T$.

c. Draw a Venn diagram with three sets, *R, S* and *T*.
Shade the area that represents $S \cap R' \cap T'$.

De Morgan's Laws

Let S and T be sets, then

$(S \cup T)' = S' \cap T'$,

$(S \cap T)' = S' \cup T'$

Example 3

Given the sets $A = \{1, 2, 4, 5, 6\}$ and $B = \{2, 3, 4, 8\}$

a. Show that $(A \cap B)' = A' \cup B'$

b. Show that $(A \cup B)' = A' \cap B'$

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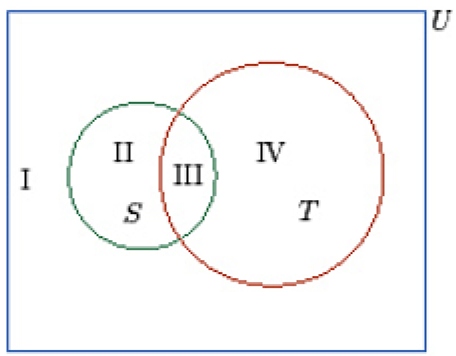
5.3 VENN DIAGRAMS AND COUNTING

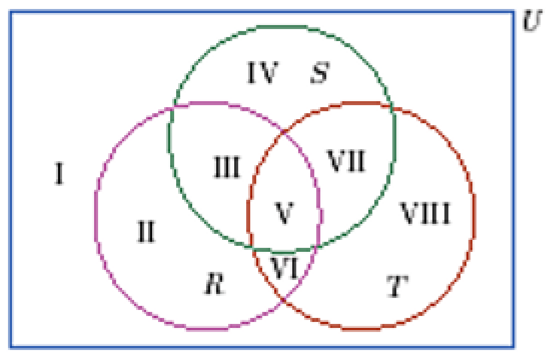
<u>Definition: Basic Regions</u>
Each Venn diagram divides the universal set into non-overlapping regions called *basic regions*.

<u>Basic regions for Venn diagram with two sets and three sets.</u>

Two sets; Regions I, II, II, IV



Three sets; Regions I, II, III, IV, V, VI, VII, VIII



<u>Example 1</u>
For the survey of the presidents of the 500 largest corporations in the US where 310 had degrees (of any sort) in business, 238 had undergraduate degrees in business, and 184 had graduate degrees in business:

a. Draw a Venn diagram and determine the number of elements in each basic region.

b. Determine the number of presidents having exactly one degree (graduate or undergraduate) in business.

Checking Your Understanding

1. Of the 1000 first-year students at a certain college, 700 take mathematics courses, 300 take mathematics and economics courses, and 200 do not take any mathematics or economics courses.

   a. Represent the data in a Venn diagram, number each region, and determine the number of elements in each basic region.

   b. How many of the first-year students take an economics course?

   c. How many take an economics course, but not a mathematics course?

Example 2

Let R, S, and T be subsets of the universal set U.
Given $n(S') = 6$, $n(S \cup T) = 10$, $n(S \cap T) = 5$, $n(T) = 7$,

a. Draw a Venn diagram to represent the information.

b. Determine the number of elements in each basic region.

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WS 5.3 Date_______________________________

1. Write the inclusion-exclusion principle.

____

2. $If\ n(S) = 6, n(R) = 9, and\ n(R \cap S) = 3, find\ n(R \cup S)$.

3. Shade a three circle Venn diagram for $T' \cap R' \cap S$.

4. A merchant surveyed 350 people to determine the way they learned about an upcoming sale. The survey showed that 200 learned about the sale from the radio. 170 from television, 160 from the newspaper, 90 from radio and television, 90 from radio and newspapers, 70 from television and newspapers, and 40 from all three sources.

a. Draw a Venn diagram to represent the given situation.
Label each region and determine and determine the number of elements in each region.

b. How many people learned of the sale from newspapers or radio, but not both?

c. How many learned of the sale only from the radio?

d. How many learned about it from the radio, but not from television?

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5.4 THE MULTIPLICATION PRINICPLE

Definition: Tree Diagram A *tree diagram* is a graph showing possibilities for an event having choices along the way.

Multiplication Principle Suppose that a task is composed of two consecutive choices. If choice 1 can be performed in m ways and, for each of these, choice 2 can be performed in n ways, then the complete task can be performed in $m \cdot n$ ways.

Example 1

Suppose a rat in a maze starts at point A. There are five possible routes to get from point A to point B and 3 possible routes to get from point B to the final destination, point C.

a. Represent the possible choices using a tree diagram.

b. Determine the solution using the multiplication principle.

Generalized Multiplication Principle

Suppose that a task consists of t choices performed consecutively. Suppose that choice 1 can be performed in m_1 ways; for each of these, choice 2 in m_2 ways; for each of these, choice 3 in m_3 ways; and so forth.

Then the task can be performed in $m_1 \cdot m_2 \cdot m_3 \ldots m_t$ ways.

Example 2

A corporation has a board of directors consisting of 10 members. The board must select from among its members a chairperson, vice chairperson, and secretary.
In how many ways can this be done?

Example 3

In how many ways can a baseball team of nine players arrange themselves in a line for a group picture?

Example 4

A certain state uses automobile license plates that consist of three letters followed by three digits. How many such license plates are there?

Example 5

There are five seats available in a sedan. In how many ways can five people be seated if only three of them can drive?

Example 6

A multiple choice exam contains 10 questions, each having 3 possible answers. How many different ways are there of completing the exam?

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5.5 PERMUTATIONS AND COMBINATIONS

<u>Definition: Permutation</u>

Given a set of n objects, a *permutation of n objects taken r at a time* is an arrangement of r of the n objects in a **<u>specific order</u>**.

The notation for a permutation is:

$P\,(n, r)$ = the number of permutations of n objects taken r at a time.

<u>Example 1</u>

How many "words" (strings of letters) of <u>two</u> distinct letters can be formed from the letters $\{a, b, c\}$?

<u>Definition: Combination</u>

Given a set of n objects, a *combination of n objects taken r at a time* is a selection of r of the n objects with **<u>order disregarded</u>**.

The notation for a combination is:

$C\,(n, r)$ = the number of combinations of n objects taken r at a time.

<u>Example 2</u>

How many two-member teams can be formed from a group that has three members $a, b,$ and c?

Formula for P(n, r)

$$P(n,1) = n$$
$$P(n,2) = n(n-1)$$
$$P(n,3) = n(n-1)(n-2)$$
$$P(n,r) = \underbrace{n(n-1)(n-2)\cdot\ldots\cdot(n-r+1)}_{r\text{ factors}}$$

Example 3

Eight horses are entered in a race in which a first, second, and third prize will be awarded. Assuming no ties, how many different outcomes are possible?

Formula for C(n, r)

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{r(r-1)\cdot\ldots\cdot 1}$$

Example 4

The board of directors of a corporation has 10 members. In how many ways can they choose a committee of 3 board members to negotiate a merger?

PRACTICE PROBLEMS

1. Calculate $P(6,3)$

2. Calculate $C(6,3)$

Determine if each example represents a permutation or a combination, then calculate.

3. A pizzeria offers five toppings for the plain cheese base of the pizzas. How many pizzas can be made that use three different toppings?

4. John couldn't recall the Serial number on his expensive bicycle. He remembered that there were 6 different digits, none used more than once, but couldn't remember what digits were used. He decided to write down all of the possible 6 digit numbers. How many different possibilities will he have to create?

5. A student must choose five courses out of seven that he would like to take. How many possibilities are there?

6. Suppose that 7 people enter a swim meet. Assuming that there are no ties, in how many ways could the gold, silver, and bronze medals be awarded?

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5.6 FURTHER COUNTING PROBLEMS

Definition: Permutation

Given a set of *n* objects, a *permutation of n objects taken r at a time* is an arrangement of *r* of the *n* objects in a **specific order**.

$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$

Definition: Combination

Given a set of *n* objects, a *combination of n objects taken r at a time* is a selection of *r* of the *n* objects with **order disregarded**.

$$C(n, r) = \frac{P(n,r)}{r!}$$

In section 5.6 we will need to recognize a permutation or combination and translate them.

Example 1 Tossing a coin 10 times
An experiment consists of tossing a coin 10 times and observing the sequence of heads and tails.

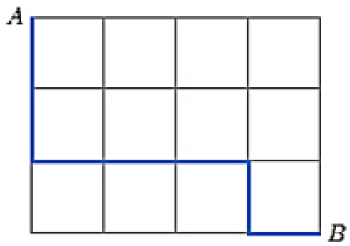
a. How many different outcomes are possible?

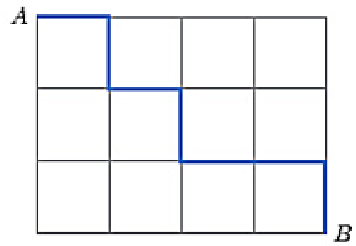
b. How many different outcomes have exactly two heads?

c. How many different outcomes have at most two heads?

d. How many different outcomes have at least two heads?

Example 2 **Routes through a City**
A tourist in a city wants to walk from point *A* to point *B* shown in the maps below.
What is the total number of routes (with no backtracking) from *A* to *B*?





Consider if S is walking a block south and E is walking a block east, the two possible routes shown in the maps could be designated as SSEEESE and ESESEES.

Example 3 **Selecting Balls from an Urn**
An urn contains 25 numbered balls, of which 15 are red and 10 are white. A sample of 5 balls is to be selected.

a. How many different samples are possible?

b. How many different samples contain all red balls?

c. How many samples contain 3 red balls and 2 white balls?

d. How many samples contain at least 4 red balls?

PRACTICE PROBLEMS

1. A bag of 10 apples contains two rotten apples and eight good apples. A shopper selects a sample of three apples from the bag.

a. How many different samples are possible?

b. How many samples contain all good apples?

c. How many samples contain at least one rotten apple?

2. A newspaper reporter wants an indication of how the 15 members of the school board feel about a certain proposal. She decides to question a sample of 6 of the board members.

a. How many samples are possible?

b. Suppose that 10 of the board members support the proposal, and 5 oppose it. How many of the samples reflect the distribution of the board? That is, in how many of the samples do 4 people support the proposal and 2 oppose it?

3. A basketball player shoots eight free throws and lists the sequence of results of each trial in order. Let S represent success and F represent failure. Then, for instance, FFSSSSSS represents the outcome of missing the first two shots and hitting the rest.

   a. How many different outcomes are possible?

   b. How many of the outcomes have six successes?

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5.7 THE BINOMIAL THEOREM

FORMULAS for C(n, r)

$$C(n,r)=\frac{P(n,r)}{r!}=\frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{r(r-1)\cdot\ldots\cdot 1}$$

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

$$C(n,r)=C(n,n-r)$$

BINOMIAL COEFFICIENT

Another notation for C(n, r) is $\binom{n}{r}$, where $\binom{n}{r}$ is called a binomial coefficient.

BINOMIAL COEFFICIENTS

$$n=2:\quad \binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$

$$n=3:\quad \binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$

PASCAL'S TRIANGLE

1	Row 0:	$\binom{0}{0}$
1 1	Row 1:	$\binom{1}{0}$ $\binom{1}{1}$
1 2 1	Row 2:	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
1 3 3 1	Row 3:	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
1 4 6 4 1	Row 4:	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$
1 5 10 10 5 1	Row 5:	$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$
1 6 15 20 15 6 1	Row 6:	$\binom{6}{0}$ $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$

Example 1 Evaluate each

a. $\binom{7}{3}$

b. $\binom{7}{4}$

BINOMIAL THEOREM

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Example 2 Expand each using the Binomial Theorem.

a. $(x+y)^4$

b. $(2x-5y)^3$

Recall from §5-1 List the possible subsets of $\{a, b, c\}$

Definition: Number of Subsets

A set with *n* elements has 2^n subsets.

Example 3 Counting Pizza Options
A pizza parlor offers a plain cheese pizza to which any number of six possible toppings can be added. How many different pizzas can be ordered?

Example 4 Book Selection
In how many ways can a selection of at least two books be made from a set of six books?